Solutions SML Math Competition Fall 2008

1. Line L has equation y = 2x + 3, and line M has the same y-intercept and is perpendicular, so M must be $y = -\frac{1}{2}x + 3$

The choices given all have a y value of 5, so we can substitute 5 for y and solve

$5 = -\frac{1}{2}x + 3$

$2 = -\frac{1}{2}x$ (Subtract 3 from both sides)

$-4 = x$ (Multiply both sides by -2)

So our answer is **A. (-4, 5)**

2. Let's declare variables, S = Sue's salary before her raise, L = Lisa's salary

$1.05S = L + 1200$ (Sue plus 5% raise is 1200 more than Lisa)

$1.01S = L$ (Lisa was 1% higher before)

$.04S = 1200$ (Subtract eq. 2 from eq. 1)

$S = 30000$ (Multiply both sides by 25 (or divide by .04))

$L = 1.01 * (30000)$ (Substitute 30000 for S in eq. 2)

$L = 30300$

So the answer is **D. $30,300**

3. If $x = -1$ is a solution than $ax^2 + bx + c$ must factor out to $(x + 1)(ax + c)$ in order to get ax^2 and c as the x^2 term and constant term. Thus the other solution must be

$ax + c = 0$

$ax = -c$ (Subtract c from both sides)

$x = -\frac{c}{a}$ (Divide both sides by a)

So the answer is **D.** $-\boldsymbol{\frac{c}{a}}$

4. 9 coins worth 45¢ can be 1 quarter, 3 nickels, and 5 pennies, or 4 dimes and 5 pennies or 9 nickels. If Ryan had said he had 5 pennies, then Sam would not know whether it was the first or second case. Therefore Ryan must have 0 pennies, and 9 nickels.

So the answer is **E. 9**

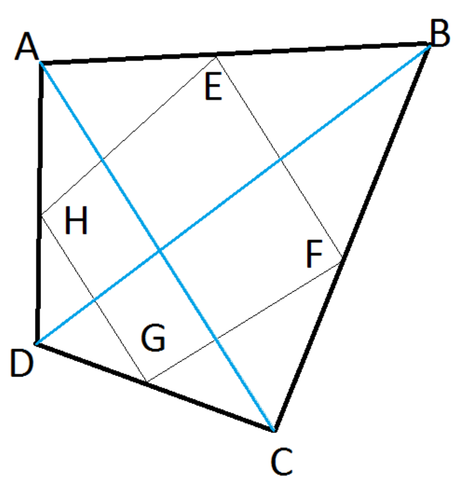
5. A lattice point will be visible if the greatest common factor of x and y is 1 (they share no factors). Of the choices given, 28 and 15 have no other common factors, (28 and 14 are both multiples of 14, 28 and 16 are both multiples of 2, 28 and 18 are multiples of 2, and 28 and 21 are multiples of 7)

So the answer is **B. (28,15)**

6. The fleas pattern is 123212321232… which can be split to 1232 1232 1232…, so each set of 4 jumps increases his position by 8. After 2008 jumps his position will have increased by $\frac{2008}{4} * 8 = 4016$. Since he is going around a clock, he will be at 4016 mod 12 (the remainder after dividing by 12), which is 8. (note: one way to do this on a calculator is to divide by 12, subtract the whole number portion, then multiply by 12)

So the answer is **E. 8**

7.



$$because\ EB\ is \frac{1}{2} of\ AB\ and\ FB\ is \frac{1}{2} of\ CB, \frac{EB}{FB} = \frac{AB}{CB}, so\ \Delta ACB\ and\ \Delta EFB\ are\ similar\ triangles. Also, \Delta ACD\ and\ \Delta HGD\ are\ similar.$$

$$Thus, EF \parallel AC \parallel HG, and\ by\ analogous\ reasoning\ GF \parallel DB \parallel HE$$

$$Thus\ EFGH\ is\ a\ parallelogram, \angle FEH = \angle FGH, and\ \angle FEH + \angle EHG = 180$$

$$EFGH\ may\ or\ may\ not\ be\ a\ square, so\ \angle FEH\ may\ or\ may\ not = \angle EHG$$

So the answer is **D. both A and C**

8. There are 16 subsets, 1 of which is empty. The sums repeat a few times but 15 sets is small enough to list:

$\{2\} = 2$	1
$\{4\} = 4$	2
$\{5\} = 5$	3
$\{2, 4\} = 6$	4
$\{2, 5\}, \{7\} = 7$	5
$\{4, 5\}, \{2, 7\} = 9$	6
$\{2, 4, 5\}, \{4, 7\} = 11$	7
$\{5, 7\} = 12$	8
$\{2, 4, 7\} = 13$	9
$\{2, 5, 7\} = 14$	10
$\{4, 5, 7\} = 16$	11
$\{2, 4, 5, 7\} = 18$	12

So the answer is **C. 12**

9. $ax - b = c$

$bx - c = a$

$ax = b + c$ (add b to both sides of eq. 1)

$bx = a + c$ (add c to both sides of eq.2)

$x = \frac{b+c}{a}$ (divide both sides of eq. 3 by a)

$x = \frac{a+c}{b}$ (divide both sides of eq.4 by b)

$\frac{a+c}{b} = \frac{b+c}{a}$ (both equivalent to x)

$a^2 + ca = b^2 + cb$ (cross-multiply)

$a^2 - b^2 = cb - ca$ (subtract b^2 and ca from both sides)

$(a - b)(a + b) = -c(a - b)$ (factor both sides)

$a + b = -c$ (divide both sides by (a-b))

$a + b + c = 0$ (add c to both sides)

So the answer is **A.** $\boldsymbol{a + b + c = 0}$

10. $\frac{x^2-22x+40}{x^2+13x-30}$

$$\frac{(x-2)(x-20)}{(x-2)(x+15)}$$

$$\frac{x-20}{x+15}$$

Vertical asymptote at x = -15

Horizontal asymptote at y = 1

Total: 2 asymptotes.

So the answer is **C. 2**

11. 55 is 5 x 11, so our number has to be a multiple of 5 and of 11. Because it is a multiple of 5, C is either 5 or 0. We want a large number, so let's start with A = 9, M=8 , and T=7. A trick for finding whether a number is divisible by 11 is if you add the odd digits (1st, 3rd, 5th digit) and subtract the even digits (2nd, 4th, 6th digits) the result will be a multiple of 11. So 9897YC is a multiple of 11 if 9+9+Y-8-7-C is a multiple of 11. this = 3 + Y - C, with C being 5 or 0. C = 0 would make Y = 8, which doesn't work because M=8 and we can't repeat. C=5 makes Y=2, so our number is 989725

Summing 9+8+9+7+2+5 = 40

An easier and faster method is to use a graphing calculator such as a TI-83 PLUS.

Use "Y=" and enter 55x. Enter 990000/55 to get a max value, use that for tablestart (2nd, window, tablestart = 18000). Then go to table (2nd, graph) and scroll up until a number in the Y column matches the AMATYC format (989725 is the highest)

So the answer is **C. 40**

12. My general method for these problems is probably not the best one, but finding integer solutions, especially with multiple variables risen to powers, is not something I am well versed in. That being said, my general approach is to guess on one variable, set the other 2 up as a Y= equation in a graphing calculator, and look in the table for integer values (or values fitting the bounds of the problem).

For this problem, a^6 is the variable that can have the least choices, so it would be the best one to guess on.

$$a^6 + b^2 + c^2 = 2009$$

$$b^2 = 2009 - a^6 - c^2$$

$$b = \sqrt{2009 - a^6 - c^2}$$

$$a = 1, b = \sqrt{2009 - 1 - c^2}$$

We can put this in the calculator as y1= $\sqrt{(}2009 - 1 - x^2)$

Now let's move to y2= and put $\sqrt{2009 - 2^6 - x^2}$

In y3, $\sqrt{2009 - 3^6 - x^2}$

4^6=4096, which is over 2009, so now we can go to our table. It should look something like this. Remember, we are looking for exactly 2 of our numbers to be powers of 2.

x	y	y2	y3
1	44.81071	44.09082	35.76311
2	44.77723	44.05678	35.72114
3	44.72136	44	35.65109
4	44.64303	43.92038	35.55278
5	44.54211	43.8178	35.42598
6	44.41846	43.6921	35.27038
7	44.27189	43.54308	35.08561
8	44.10215	43.3705	34.87119
9	43.909	43.17407	34.62658
10	43.6921	42.95346	34.35113
11	43.45112	42.70831	34.04409
12	43.18565	42.43819	33.7046
13	42.89522	42.14262	33.33167
14	42.57934	41.82105	32.92416
15	42.23742	41.47288	32.48076
16	41.86884	41.09745	32
17	41.47288	40.69398	31.48015
18	41.04875	40.26164	30.91925
19	40.59557	39.7995	30.31501
20	40.11234	39.30649	29.66479
21	39.59798	38.78144	28.9655
22	39.05125	38.22303	28.21347
23	38.47077	37.62978	27.40438
24	37.85499	37	26.533
25	37.20215	36.3318	25.59297

26	36.51027	35.62303	24.57641
27	35.77709	34.87119	23.47339
28	35	34.07345	22.27106
29	34.17601	33.2265	20.95233
30	33.30165	32.32646	19.49359
31	32.37283	31.36877	17.86057
32	31.38471	30.34798	16
33	30.3315	29.25748	13.82027
34	29.20616	28.08914	11.13553
35	28	26.83282	7.416198
36	26.70206	25.47548	#NUM!
37	25.29822	24	#NUM!
38	23.76973	22.38303	#NUM!
39	22.09072	20.59126	#NUM!
40	20.22375	18.57418	#NUM!
41	18.11077	16.24808	#NUM!
42	15.65248	13.45362	#NUM!
43	12.64911	9.797959	#NUM!
44	8.544004	3	#NUM!
45	#NUM!	#NUM!	#NUM!

Note the highlighted cell, in which a = 3, b = 16, and c = 32

16 and 32 are both powers of 2, so that is the answer we are looking for.

16 + 32 + 3 = 51

So the answer is **E. 51**

13. 3 paydays in February can only happen in a leap year that starts as a payday (February 1, 15, and 29 being paydays). So we are looking for some multiple of 4 years (leap years are 4 years apart) that will also be a multiple of 14 days (they get paid every 14 days). First off, there are 365 days in 3 of the years and 366 in the 4th, so 4 years = 1461 days, which is 5 days more than a multiple of 14. 5 and 14 don't share any factors, so it will take 14 sets of 4 years to cycle back to having a payday on Feb 1st of a leap year. 14 * 4 = 56 years. 56 + 2008 = 2064

So the answer is **C. 4**

14. One way to do this is to look at each possibility and see who is lying, then count them up:

	If A did it	B did it	C did it	D did it	E did it
A	FALSE	FALSE	FALSE	TRUE	FALSE
B	TRUE	FALSE	TRUE	TRUE	TRUE

C	TRUE	TRUE	TRUE	TRUE	FALSE
D	TRUE	TRUE	TRUE	FALSE	TRUE
E	TRUE	FALSE	TRUE	TRUE	TRUE
# true	4	2	4	4	3
# false	1	3	1	1	2

Showing that E did it.

Another way to reason is:

A and D provide contradictory statements, so 1 will always be false and 1 will always be true.

Therefore, between B, C, and E, exactly 1 is lying.

B and E say the same thing, so either both are lying or both are telling the truth.

They must both be telling the truth.

So C must be lying.

Therefore E must be guilty.

So the answer is **E. E**

15. let's label the 2 arithmetic sequences as $x + kn$ and $y + cn$, with n being the number of the term, starting at 0.

n=0 $\qquad xy = 468$

n=1 $\qquad (x + k)(y + c) = 462$

$xy + xc + ky + kc = 462$

n=2 $\qquad (x + 2k)(y + 2c) = 384$

$xy + 2xc + 2ky + 4kc = 384$

We want $(x + 3k)(y + 3c)$, which is equivalent to $xy + 3xc + 3ky + 9kc$

We can set this up as a system of equations:

(a)	$xy = 468$	given
(b)	$xy + (xc + ky) + kc = 462$	given
(c)	$xy + 2(xc + ky) + 4kc = 384$	given
(d)	$(xc + ky) + kc = -6$	(b)-(a)

(e)	$2(xc + ky) + 4kc = -84$	(c)-(a)
(f)	$2kc = -72$	(e)-2(d)
(g)	$kc = -36$	(f)/2
(h)	$xc + ky = 30$	(d)-(g)
(i)	$xy + 3(xc + ky) + 9kc = 234$	(a)+3(h)+9(g)

So the answer is **234**, which is not available. So AMATYC gave everyone that question for free.

16. First we need to change $\frac{\tan(\frac{A-B}{2})}{\tan(\frac{A+B}{2})}$ to something more sane.

(a) $$\frac{\tan(\frac{A-B}{2})}{\tan(\frac{A+B}{2})}$$

(b) $$\frac{\frac{\sin(\frac{A}{2}-\frac{B}{2})}{\cos(\frac{A}{2}-\frac{B}{2})}}{\frac{\sin(\frac{A}{2}+\frac{B}{2})}{\cos(\frac{A}{2}+\frac{B}{2})}}$$ $Substitution\ using\ \tan(x) = \frac{\sin(x)}{\cos(x)}$

(c) $$\frac{\sin\left(\frac{A}{2}-\frac{B}{2}\right)}{\cos\left(\frac{A}{2}-\frac{B}{2}\right)} * \frac{\cos\left(\frac{A}{2}+\frac{B}{2}\right)}{\sin\left(\frac{A}{2}+\frac{B}{2}\right)}$$ $flip\ and\ multiply$

(d) $$\frac{(\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)-\sin\left(\frac{B}{2}\right)\cos(\frac{A}{2}))(\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)-\sin\left(\frac{A}{2}\right)\sin(\frac{B}{2}))}{(\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)+\sin\left(\frac{A}{2}\right)\sin(\frac{B}{2}))(\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)+\sin\left(\frac{B}{2}\right)\cos(\frac{A}{2}))}$$ angle sum identities

(e) $$\frac{\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)\cos^2\left(\frac{B}{2}\right)-\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)\sin^2\left(\frac{A}{2}\right)-\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)\cos^2\left(\frac{A}{2}\right)+\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)\sin^2(\frac{B}{2})}{\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)\cos^2\left(\frac{B}{2}\right)+\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)\cos^2\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)\sin^2\left(\frac{A}{2}\right)+\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)\sin^2(\frac{B}{2})}$$

(f) $$\frac{\sin\left(\frac{A}{2}\right)\cos(\frac{A}{2})(\sin^2\left(\frac{B}{2}\right)+\cos^2(\frac{B}{2})) - \sin(\frac{B}{2})\cos(\frac{B}{2})(\sin^2\left(\frac{A}{2}\right)+\cos^2(\frac{A}{2}))}{\sin\left(\frac{A}{2}\right)\cos(\frac{A}{2})(\sin^2\left(\frac{B}{2}\right)+\cos^2(\frac{B}{2})) + \sin(\frac{B}{2})\cos(\frac{B}{2})(\sin^2\left(\frac{A}{2}\right)+\cos^2(\frac{A}{2}))}$$

(g) $$\frac{\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)-\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)}{\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)}$$ $\sin^2 a + \cos^2 a = 1$

(h) $$\frac{\frac{\sin(A)}{2}-\frac{\sin(B)}{2}}{\frac{\sin(A)}{2}+\frac{\sin(B)}{2}}$$ $2\sin(a)\cos(a) = \sin(2a)$

(i) $$\frac{\sin(A)-\sin(B)}{\sin(A)+\sin(B)}$$ multiply by 2/2

Now we can use the law of sines and substitution.

(j) $\frac{\sin(A)}{9} = \frac{\sin(B)}{7}$ law of sines

(k) $\sin(B) = \frac{7\sin(A)}{9}$

(l) $\frac{\sin(A) - \frac{7\sin(A)}{9}}{\sin(A) + \frac{7\sin(A)}{9}}$ substitution (k) into (i)

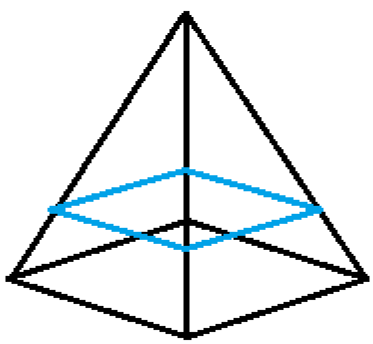
(m) $\frac{9\sin(A) - 7\sin(A)}{9\sin(A) + 7\sin(A)}$ multiply by 9/9

(n) $\frac{2\sin(A)}{16\sin(A)}$

(o) $\frac{1}{8}$ reduce

So the answer is A. $\frac{1}{8}$

17.



The equation for volume of a pyramid is $A = \frac{1}{3} base * height$

The original pyramid has a square base, $6 * 6 = 36$, and a height of 9

$$\frac{1}{3} * 9 * 36 = 108$$

The upper 6 m of the pyramid also have a square base but we have to find the length ourselves.

The proportion of height to base should be the same in both pyramids, though.

$$\frac{9}{6} = \frac{6}{x}$$

$$36 = 9x$$

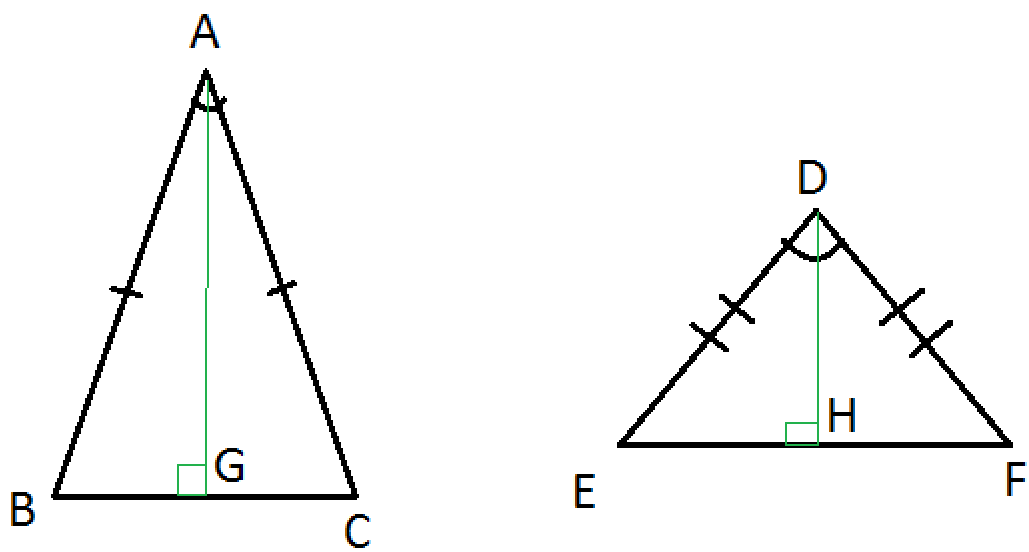
$$x = 4$$

So the top 6 m have volume $= 4 * 4 * \frac{1}{3} * 6 = 32$

108-32=72=the volume of the bottom 3 m

So the answer is D. 72 m^3

18.



The area of a triangle $= \frac{1}{2}bh$

For $\Delta ABC, the\ area\ = \frac{1}{2}BC * AG$

Which in terms of A and AB $= AB * sin\left(\frac{A}{2}\right) * AB * \cos\left(\frac{A}{2}\right)$

The area of $\Delta DEF\ = DE * \sin\left(\frac{D}{2}\right) * DE * \cos(\frac{D}{2})$

$=\frac{AB}{2} * \sin(A) * \frac{AB}{2} * \cos(A)$

So the ratio of $\Delta ABC\ to\ \Delta DEF\ = \frac{AB^2 \sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}{\frac{AB^2}{4}\sin(A)\cos(A)}$

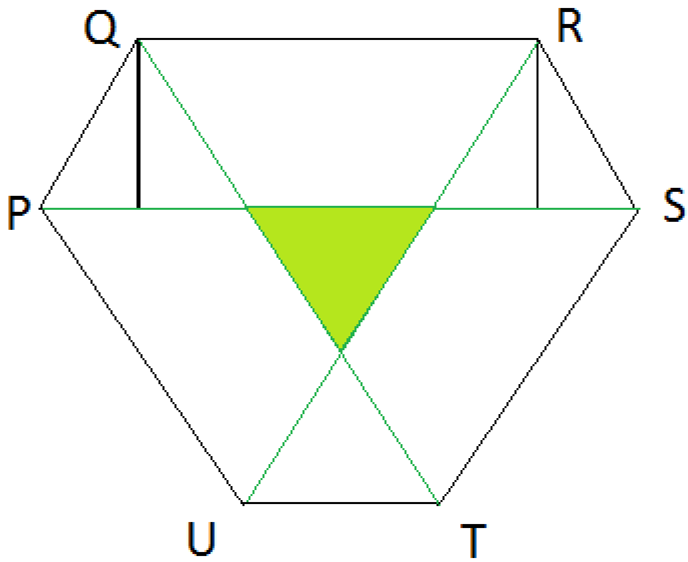
$=\frac{\frac{\sin(A)}{2}}{\frac{1}{4}\sin(A)\cos(A)}$

$=\frac{2}{\cos(A)}$

$=2\sec(A)$

So the answer is B. 2 sec(A)

19.



We want the area of the green triangle in the center.

$RS\ goes\ down\ the\ same\ amount\ as\ QP, so\ PS\ and\ QR\ are\ parallel.$

$Thus, \angle PSR + \angle SRQ = 180. \angle QRS = 120, so\ \angle PRS = 60.$

$If\ we\ draw\ a\ veritical\ line\ from\ R\ to\ PS, and\ another$

$from\ Q\ to\ PS, we\ make\ two\ 30{,}60{,}90\ right\ triangles\ and\ a\ rectangle.$

$PD = SE = 50\cos(60) = 25, and\ DE = QR = 100$

$PS = PD + DE + ES = 25 + 100 + 25 = 150$

$DA = PD = BE = 25, so\ AB = 50$

$\angle ABC = \angle BCA = \angle CAB = 60°$

$\Delta ABC = \frac{1}{2} * 50 * 50\sin(60) = 1250\sin(60) = 1082.5317$

So the answer is A. 1082.5

20.

$(2^{2^{k+1}} + 2^{2^k} + 1)(2^{2^{k+1}} - 2^{2^k} + 1)$

$(\left(2^{(2)2^k} + 1\right) + 2^{2^k})(\left(2^{(2)2^k} + 1\right) - 2^{2^k})$

$\left(2^{(2)2^k}+1\right)^2-\left(2^{2^k}\right)^2$ (difference of 2 perfect squares)

$2^{(4)2^k}+(2)2^{(2)2^k}+1-2^{(2)2^k}$

$2^{(4)2^k}+2^{(2)2^k}+1$

$2^{2^{k+2}}+2^{2^{k+1}}+1$

When k=0, we have the first 2 terms in the sequence (that is, $(2^2+2+1)(2^2-2+1)$)

(and the -1 at the end, but we'll leave that for now), which resolves to

$(2^{2^2}+2^2+1)$ (as shown above). That combines with the term added when k=1 to make $(2^{2^3}+2^{2^2}+1)$, which combines with the term added when k=2 in similar fashion, and so on. So at any point we have $\left(2^{2^{k+2}}+2^{2^{k+1}}+1\right)-1$, which reduces to

$\left(2^{2^{k+2}}+2^{2^{k+1}}\right)$

$((2^{2^{k+1}})^2+2^{2^{k+1}})$

$2^{2^{k+1}}(2^{2^{k+1}}+1)$

So n=$2^{2^{k+1}}$ and we just need to figure out when $2^{2^{k+1}}+2^{2^{k+1}}+1\geq 10^{1000}$

Unfortunately, the TI-83 or similar doesn't handle numbers that large. We can fix that using logs, however. I don't include the 1 because of how little it matters with numbers this large.

$2\left(2^{2^{k+1}}\right)\geq 10^{1000}$

$2^{2^{k+1}+1}\geq 10^{1000}$

$\log(2^{2^{k+1}+1})\geq 1000$ log both sides

$(2^{k+1}+1)\log(2)\geq 1000$

$2^{k+1}-1\geq \frac{1000}{\log(2)}-1$

$2^{k+1}\geq 3321$

$2^{9+1}=1024$

$2^{10+1}=2048$

$2^{11+1}=4096\geq 3322$

So the answer is **C. 11**